CONCENTRATION OF 1-LIPSCHITZ MAPS INTO AN INFINITE DIMENSIONAL ℓ^p-BALL WITH THE ℓ^q-DISTANCE FUNCTION

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Abstract. In this paper, we study the Lévy-Milman concentration phenomenon of 1-Lipschitz maps into infinite dimensional metric spaces. Our main theorem asserts that the concentration to an infinite dimensional ℓ^p-ball with the ℓ^q-distance function for $1 \leq p < q \leq +\infty$ is equivalent to the concentration to the real line.

1. Introduction

This paper is devoted to investigating the Lévy-Milman concentration phenomenon of 1-Lipschitz maps from mm-spaces (metric measure spaces) to infinite dimensional metric spaces. Here, an *mm-space* is a triple (X, d_X, μ_X), where d_X is a complete separable metric on a set X and μ_X a finite Borel measure on (X, d_X). The theory of concentration of 1-Lipschitz functions was first introduced by V. D. Milman in his investigation of asymptotic geometric analysis ([17], [18], [19]). Nowadays, the theory blend with various areas of mathematics, such as geometry, functional analysis and infinite dimensional integration, discrete mathematics and complexity theory, probability theory, and so on (see [16], [21], [22], [24] and the references therein for further information).

The theory of concentration of maps into general metric spaces was first studied by M. Gromov ([11], [12], [13]). He established the theory by introducing the observable diameter $\mathrm{ObsDiam}_Y(X; -\kappa)$ for an mm-space X, a metric space Y, and $\kappa > 0$ in [13] (see Section 2 for the definition of the observable diameter). Given a sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces and a metric space Y, we note that $\lim_{n\to\infty} \mathrm{ObsDiam}_Y(X_n; -\kappa) = 0$ for any $\kappa > 0$ if and only if for any sequence $\{f_n : X_n \to Y\}_{n=1}^\infty$ of 1-Lipschitz maps, there exists a sequence $\{m_{f_n}\}_{n=1}^\infty$ of points in Y such that

$$\lim_{n\to\infty} \mu_{X_n}(\{x_n \in X_n \mid d_Y(f_n(x_n), m_{f_n}) \geq \varepsilon\}) = 0$$

for any $\varepsilon > 0$. If $\lim_{n\to\infty} \mathrm{ObsDiam}_{\mathbb{R}}(X_n; -\kappa) = 0$ for any $\kappa > 0$, then the sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces is called a *Lévy family*. The Lévy families were first introduced and analyzed by Gromov and Milman in [10]. In our previous works [2], [3], [4], [5], the

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author proved that if a metric space Y is either an $\mathbb{R}$-tree, a doubling space, a metric graph, or a Hadamard manifold, then $\lim_{n\to\infty} \mathrm{ObsDiam}_Y(X_n; -\kappa) = 0$ holds for any $\kappa > 0$ and any Lévy family $\{X_n\}_{n=1}^{\infty}$. To prove these results, we needed to assume the finiteness of the dimension of the target metric spaces.

In this paper, we treat the case where the dimension of the target metric space Y is infinite. The author has proved in [1] that if the target space Y is so big that an mm-space X with some homogeneity property can isometrically be embedded into Y, then its observable diameter $\mathrm{ObsDiam}_Y(X; -\kappa)$ is not close to zero. It seems from this result that the concentration to an infinite dimensional metric space cannot happen easily.

A main theorem of this paper is the following. For $1 \leq p \leq +\infty$, we denote by $B_{\ell^p}^{\infty}$ an infinite dimensional ℓ^p-ball $\{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\infty} \mid \sum_{n=1}^{\infty} |x_n|^p \leq 1\}$ and by d_{ℓ^p} the ℓ^p-distance function.

Theorem 1.1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of mm-spaces and $1 \leq p < q \leq +\infty$. Then, the sequence $\{X_n\}_{n=1}^{\infty}$ is a Lévy family if and only if*

$$\lim_{n\to\infty} \mathrm{ObsDiam}_{(B_{\ell^p}^{\infty}, d_{\ell^q})}(X_n; -\kappa) = 0 \text{ for any } \kappa > 0. \tag{1.1}$$

As a result, we obtain the example of the infinite dimensional target metric space such that the concentration to the space happens as often as the concentration to the real line.

The proof of the sufficiency of Theorem 1.1 is easy. A. Gournay and M. Tsukamoto's observations play important roles for the proof of the converse ([9], [28]). Answering a question of Gromov in [14, Section 1.1.4], Tsukamoto proved in [28] that the "macroscopic" dimension of the space $(B_{\ell^p}^{\infty}, d_{\ell^q})$ for $1 \leq p < q \leq +\infty$ is finite. Gournay independently proved it in [9] in the case of $q = +\infty$. For any p and q with $1 \leq q \leq p \leq +\infty$, we have an example of a Lévy family which does not satisfy (1.1) (see Proposition 4.4).

As applications of Theorem 1.1, by virtue of [3, Propositions 4.3 and 4.4], we obtain the following corollaries of a Lévy group action. A Lévy group was first introduced by Gromov and Milman in [10]. Let a topological group G acts on a metric space X. The action is called *bounded* if for any $\varepsilon > 0$ there exists a neighborhood U of the identity element $e_G \in G$ such that $d_X(x, gx) < \varepsilon$ for any $g \in U$ and $x \in X$. Note that every bounded action is continuous. We say that the topological group G acts on X *by uniform isomorphisms* if for each $g \in G$, the map $X \ni x \mapsto gx \in X$ is uniform continuous. The action is said to be *uniformly equicontinuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(gx, gy) < \varepsilon$ for every $g \in G$ and $x, y \in X$ with $d_X(x, y) < \delta$. Given a subset $S \subseteq G$ and $x \in X$, we put $Sx := \{gx \mid g \in S\}$.

Corollary 1.2. *Let $1 \leq p < q \leq +\infty$ and assume that a Lévy group G boundedly acts on the metric space $(B_{\ell^p}^{\infty}, d_{\ell^q})$ by uniform isomorphisms. Then for any compact subset $K \subseteq G$ and any $\varepsilon > 0$, there exists a point $x_{\varepsilon,K} \in B_{\ell^p}^{\infty}$ such that* $\mathrm{diam}(Kx_{\varepsilon,K}) \leq \varepsilon$.

Corollary 1.3. *There are no non-trivial bounded uniformly equicontinuous actions of a Lévy group to the metric space $(B_{\ell^p}^{\infty}, d_{\ell^q})$ for $1 \leq p < q \leq +\infty$.*

Gromov and Milman pointed out in [10] that the unitary group $U(\ell^2)$ of the separable Hilbert space ℓ^2 with the strong topology is a Lévy group. Many concrete examples

of Lévy groups are known by the works of S. Glasner [8], H. Furstenberg and B. Weiss (unpublished), T. Giordano and V. Pestov [6], [7], and Pestov [25], [26]. For examples, groups of measurable maps from the standard Lebesgue measure space to compact groups, unitary groups of some von Neumann algebras, groups of measure and measure-class preserving automorphisms of the standard Lebesgue measure space, full groups of amenable equivalence relations, and the isometry groups of the universal Urysohn metric spaces are Lévy groups (see the recent monograph [24] for precise).

2. Preliminaries

Let Y be a metric space and ν a Borel measure on Y such that $m := \nu(Y) < +\infty$. We define for any $\kappa > 0$

$$\operatorname{diam}(\nu, m-\kappa) := \inf\{\operatorname{diam} Y_0 \mid Y_0 \subseteq Y \text{ is a Borel subset such that } \nu(Y_0) \geq m-\kappa\}$$

and call it the *partial diameter* of ν.

Definition 2.1 (Observable diameter). Let (X, d_X, μ_X) be an mm-space with $m_X := \mu_X(X)$ and Y a metric space. For any $\kappa > 0$ we define the *observable diameter* of X by

$$\operatorname{ObsDiam}_Y(X; -\kappa) := \sup\{\operatorname{diam}(f_*(\mu_X), m_X - \kappa) \mid f : X \to Y \text{ is a 1-Lipschitz map}\},$$

where $f_*(\mu_X)$ stands for the push-forward measure of μ_X by f.

The idea of the observable diameter comes from the quantum and statistical mechanics, that is, we think of μ_X as a state on a configuration space X and f is interpreted as an observable.

Let (X, d_X, μ_X) be an mm-space. For any $\kappa_1, \kappa_2 \geq 0$, we define the *separation distance* $\operatorname{Sep}(X; \kappa_1, \kappa_2) = \operatorname{Sep}(\mu_X; \kappa_1, \kappa_2)$ of X as the supremum of the distance $d_X(A, B) := \inf\{d_X(a, b) \mid a \in A \text{ and } b \in B\}$, where A and B are Borel subsets of X satisfying that $\mu_X(A) \geq \kappa_1$ and $\mu_X(B) \geq \kappa_2$.

Lemma 2.2 (cf. [13, Section $3\frac{1}{2}$.33]). *Let X and Y be two mm-spaces and $\alpha > 0$. Assume that an α-Lipschitz map $f : X \to Y$ satisfies $f_*(\mu_X) = \mu_Y$. Then we have*

$$\operatorname{Sep}(Y; \kappa_1, \kappa_2) \leq \alpha \operatorname{Sep}(X; \kappa_1, \kappa_2).$$

Relationships between the observable diameter and the separation distance are followings. We refer to [4, Subsection 2.2] for precise proofs.

Lemma 2.3 (cf. [13, Section $3\frac{1}{2}$.33]). *Let X be an mm-space and $\kappa, \kappa' > 0$ with $\kappa > \kappa'$. Then we have*

$$\operatorname{ObsDiam}_{\mathbb{R}}(X; -\kappa') \geq \operatorname{Sep}(X; \kappa, \kappa).$$

Remark 2.4. In [13, Section $3\frac{1}{2}$.33], Lemma 2.3 is stated as $\kappa = \kappa'$, but that is not true in general. For example, let $X := \{x_1, x_2\}$, $d_X(x_1, x_2) := 1$, and $\mu_X(\{x_1\}) = \mu_X(\{x_2\}) := 1/2$. Putting $\kappa = \kappa' = 1/2$, we have $\operatorname{ObsDiam}_{\mathbb{R}}(X; -1/2) = 0$ and $\operatorname{Sep}(X; 1/2, 1/2) = 1$.

Lemma 2.5 (cf. [13, Section $3\frac{1}{2}.33$]). *Let ν be a Borel measure on $\mathbb{R}$ with $m := \nu(\mathbb{R}) < +\infty$. Then, for any $\kappa > 0$ we have*

$$\operatorname{diam}(\nu, m - 2\kappa) \leq \operatorname{Sep}(\nu; \kappa, \kappa).$$

In particular, for any $\kappa > 0$ we have

$$\operatorname{ObsDiam}_{\mathbb{R}}(X; -2\kappa) \leq \operatorname{Sep}(X; \kappa, \kappa).$$

Combining Lemma 2.3 with Lemma 2.5, we obtain the following corollary:

Corollary 2.6 (cf. [13, Section $3\frac{1}{2}.33$]). *A sequence $\{X_n\}_{n=1}^{\infty}$ of mm-spaces is a Lévy family if and only if $\lim_{n\to\infty} \operatorname{Sep}(X_n; \kappa, \kappa) = 0$ for any $\kappa > 0$.*

Lemma 2.7. *Let ν be a finite Borel measure on $(\mathbb{R}^k, d_{\ell^p})$ with $m := \nu(\mathbb{R}^k)$. Then for any $\kappa > 0$ we have*

$$\operatorname{diam}(\nu, m - \kappa) \leq k^{1/p} \operatorname{Sep}\Big(\nu; \frac{\kappa}{2k}, \frac{\kappa}{2k}\Big).$$

Proof. For $i = 1, 2, \cdots, k$, let $\operatorname{pr}_i : \mathbb{R}^k \ni (x_i)_{i=1}^k \mapsto x_i \in \mathbb{R}$ be the projection. For Borel subsets $A_1, A_2, \cdots, A_k \subseteq \mathbb{R}$ with $(\operatorname{pr}_i)_*(\nu)(A_i) \geq \kappa/k$, we have

$$\nu(A_1 \times A_2 \times \cdots \times A_k) = \nu\Big(\bigcap_{i=1}^{k} (\operatorname{pr}_i)^{-1}(A_i)\Big) \geq m - \kappa,$$

which leads to

$$\operatorname{diam}(\nu, m - \kappa) \leq \operatorname{diam}(A_1 \times A_2 \times \cdots \times A_k) \leq k^{1/p} \max_{1 \leq i \leq k} \operatorname{diam} A_i.$$

We therefore get

$$\operatorname{diam}(\nu, m - \kappa) \leq k^{1/p} \max_{1 \leq i \leq k} \operatorname{diam}\Big((\operatorname{pr}_i)_*(\nu), m - \frac{\kappa}{k}\Big).$$

Combining this with Lemmas 2.2 and 2.5, we obtain

$$\operatorname{diam}(\nu, m - \kappa) \leq k^{1/p} \max_{1 \leq i \leq k} \operatorname{Sep}\Big((\operatorname{pr}_i)_*(\nu); \frac{\kappa}{2k}, \frac{\kappa}{2k}\Big) \leq k^{1/p} \operatorname{Sep}\Big(\nu; \frac{\kappa}{2k}, \frac{\kappa}{2k}\Big).$$

This completes the proof. □

Lemma 2.8. *Let a, b be two real numbers with $a < b$. Then, a sequence $\{X_n\}_{n=1}^{\infty}$ of mm-spaces is a Lévy family if and only if*

$$\lim_{n\to\infty} \operatorname{ObsDiam}_{[a,b]}(X_n; -\kappa) = 0 \text{ for any } \kappa > 0. \tag{2.1}$$

Proof. The necessity is obvious. We shall prove the converse. Suppose that the sequence $\{X_n\}_{n=1}^{\infty}$ with the property (2.1) is not a Lévy family. Then, by Corollary 2.6, there exists $\kappa > 0$ and Borel subsets $A_n, B_n \subseteq X_n$ such that $\mu_{X_n}(A_n) \geq \kappa$, $\mu_{X_n}(B_n) \geq \kappa$, and $\limsup_{n\to\infty} d_{X_n}(A_n, B_n) > 0$. Define a function $f_n : X_n \to \mathbb{R}$ by $f_n(x) := \max\{d_{X_n}(x, A_n) + a, b\}$. Since $\mu_{X_n}(B_n) \geq \kappa$ and $\limsup_{n\to\infty} d_{X_n}(A_n, B_n) > 0$, we have

$$\limsup_{n\to\infty} \operatorname{diam}((f_n)_*(\mu_{X_n}), m_{X_n} - \kappa') > 0$$

for any $0 < \kappa' < \kappa$. Since each f_n is a 1-Lipschitz function, this contradicts the assumption (2.1). This completes the proof. □

3. Proof of the main theorem

To prove the main theorem, we extract from Gournay's paper [9] and Tsukamoto's paper [28] their arguments.

For $k \in \mathbb{N}$, we identify $\mathbb{R}^k$ with the subset $\{(x_1, x_2, \cdots, x_k, 0, 0, \cdots) \in \mathbb{R}^\infty \mid x_i \in \mathbb{R} \text{ for all } i\}$ of $\mathbb{R}^\infty$. Given $k \in \mathbb{N} \cup \{\infty\}$, let $\mathfrak{S}_k$ be the k-th symmetric group. We consider the group $G_k := \{\pm 1\}^k \rtimes \mathfrak{S}_k$. The multiplication in G_k is given by

$$((\varepsilon_n)_{n=1}^k, \sigma) \cdot ((\varepsilon'_n)_{n=1}^k, \sigma') := ((\varepsilon_n \varepsilon'_{\sigma^{-1}(n)})_{n=1}^k, \sigma\sigma').$$

The group G_k acts on the space $\mathbb{R}^k$ by

$$((\varepsilon_n)_{n=1}^k, \sigma) \cdot (x_n)_{n=1}^k := (\varepsilon_n x_{\sigma^{-1}(n)})_{n=1}^k.$$

Note that this action preserves the k-dimensional ℓ^p-ball $B^k_{\ell^p} \subseteq B^\infty_{\ell^p}$ and the ℓ^q-distance function d_{ℓ^q}. Define a subset $\Lambda_k \subseteq B^k_{\ell^p}$ by

$$\Lambda_k := \{x \in B^k_{\ell^p} \mid x_{i-1} \geq x_i \geq 0 \text{ for all } i\}.$$

Given an arbitrary $\varepsilon > 0$, we put $k(\varepsilon) := \lceil (2/\varepsilon)^{pq/(q-p)} \rceil - 1$, where $\lceil (2/\varepsilon)^{pq/(q-p)} \rceil$ denotes the smallest integer which is not less than $(2/\varepsilon)^{pq/(q-p)}$. For $k \geq k(\varepsilon) + 1$, we define a continuous map $f_{k,\varepsilon} : \Lambda_k \to \mathbb{R}^{k(\varepsilon)}$ by

$$f_{k,\varepsilon}(x) := (x_1 - x_{k(\varepsilon)+1}, x_2 - x_{k(\varepsilon)+1}, \cdots, x_{k(\varepsilon)} - x_{k(\varepsilon)+1}, 0, 0, \cdots).$$

For any $x \in B^k_{\ell^p}$, taking $g \in G_k$ such that $gx \in \Lambda_k$, we define

$$F_{k,\varepsilon}(x) := g^{-1} f_{k,\varepsilon}(gx).$$

This definition of the map $F_{k,\varepsilon} : B^k_{\ell^p} \to B^k_{\ell^p}$ is well-defined (see [28, Section 2] for details). Given $k \in \mathbb{N}$, we put $A_k := \bigcup_{g \in G_\infty} g\mathbb{R}^k \subseteq \mathbb{R}^\infty$.

Theorem 3.1 (cf. [9, Proposition 1.3] and [28, Section 2])**.** *The map $F_{k,\varepsilon} : B^k_{\ell^p} \to B^k_{\ell^p}$ satisfies that $F_{k,\varepsilon}(B^k_{\ell^p}) \subseteq A_{k(\varepsilon)}$ and*

$$d_{\ell^q}(x, F_{k,\varepsilon}(x)) \leq \frac{\varepsilon}{2} \tag{3.1}$$

for any $x \in B^k_{\ell^p}$.

Lemma 3.2. *The map $F_{k,\varepsilon} : (B^k_{\ell^p}, d_{\ell^q}) \to (A_{k(\varepsilon)}, d_{\ell^q})$ is a $(1 + k(\varepsilon)^{1/q})$-Lipschitz map.*

Proof. By the definition of the map $F_{k,\varepsilon}$, it suffices to prove that the map $F := F_{2k(\varepsilon)+2,\varepsilon} : (B^{2k(\varepsilon)+2}_{\ell^p}, d_{\ell^q}) \to (B^{2k(\varepsilon)+2}_{\ell^p}, d_{\ell^q})$ is $(1 + k(\varepsilon)^{1/q})$-Lipschitz. Recall that

$$F(x) = (x_1 - x_{k(\varepsilon)+1}, x_2 - x_{k(\varepsilon)+1}, \cdots, x_{k(\varepsilon)} - x_{k(\varepsilon)+1}, 0, 0, \cdots, 0)$$

for any $x \in \Lambda_{2k(\varepsilon)+2}$. We hence get

$$d_{\ell^q}(F(x), F(y)) \leq d_{\ell^q}(x, y) + k(\varepsilon)^{1/q} |x_{k(\varepsilon)+1} - y_{k(\varepsilon)+1}| \leq (1 + k(\varepsilon)^{1/q})\, d_{\ell^q}(x, y)$$

for any $x, y \in \Lambda_{2k(\varepsilon)+2}$. Since each $g \in G_{2k(\varepsilon)+2}$ preserves the distance function d_{ℓ^q}, the map F is $(1+k(\varepsilon)^{1/q})$-Lipschitz on each $g\Lambda_{2k(\varepsilon)+2}$.

Let $x, y \in B_{\ell^p}^{2k(\varepsilon)+2}$ be arbitrary points. Observe that there exist $t_0 := 0 \leq t_1 \leq t_2 \leq \cdots \leq t_{i-1} \leq 1 =: t_i$ and $g_1, g_2, \cdots, g_i \in G_{2k(\varepsilon)+2}$ such that $(1-t)x + ty \in g_j\Lambda_{2k(\varepsilon)+2}$ for any $t \in [t_{j-1}, t_j]$. We therefore obtain

$$\begin{aligned} d_{\ell^q}(F(x), F(y)) &\leq \sum_{j=1}^{i} d_{\ell^q}(F((1-t_{j-1})x + t_{j-1}y), F((1-t_j)x + t_j y)) \\ &\leq (1+k(\varepsilon)^{1/q}) \sum_{j=1}^{i} d_{\ell^q}((1-t_{j-1})x + t_{j-1}y, (1-t_j)x + t_j y) \\ &= (1+k(\varepsilon)^{1/q})\, d_{\ell^q}(x, y). \end{aligned}$$

This completes the proof. ☐

The following lemma is a key to prove Theorem 1.1.

Lemma 3.3. *Let $k \in \mathbb{N}$ and $\{\nu_{n,k}\}_{n=1}^{\infty}$ be a sequence of finite Borel measures on (A_k, d_{ℓ^q}) satisfying that*

$$\lim_{n\to\infty} \mathrm{Sep}(\nu_{n,k}; \kappa_1, \kappa_2) = 0 \tag{3.2}$$

for any $\kappa_1, \kappa_2 > 0$. Then, putting $m_n := \nu_{n,k}(A_k)$, we have

$$\lim_{n\to\infty} \mathrm{diam}(\nu_{n,k}, m_n - \kappa) = 0 \tag{3.3}$$

for any $\kappa > 0$.

Proof. It suffices to prove (3.3) by choosing a subsequence. We shall prove it by induction for k.

For $k = 0$, since $A_0 = \{(0, 0, \cdots)\}$, we have $\mathrm{diam}(\nu_{n,0}, m_n - \kappa) = 0$.

Assume that (3.3) holds for any sequence $\{\nu_{n,k-1}\}_{n=1}^{\infty}$ of finite Borel measures on (A_{k-1}, d_{ℓ^q}) having the property (3.2). Let $\{\nu_{n,k}\}_{n=1}^{\infty}$ be any sequence of finite Borel measures on (A_k, d_{ℓ^q}) having the property (3.2). Since $\lim_{n\to\infty} m_n = 0$ implies (3.3), we assume that $\inf_{n\in\mathbb{N}} m_n > 0$. Putting

$$a_n := \max\Big\{ \mathrm{Sep}\Big(\nu_{n,k}; \frac{m_n}{6}, \frac{\kappa}{2}\Big), \mathrm{Sep}\Big(\nu_{n,k}; \frac{m_n}{6}, \frac{m_n}{6}\Big)\Big\},$$

we get $\lim_{n\to\infty} a_n = 0$ by the assumption (3.2) and $\inf_{n\in\mathbb{N}} m_n > 0$. Define subsets $B_{n,1}$ and $B_{n,2}$ of the set A_k by $B_{n,1} := (A_{k-1})_{a_n} \cap A_k$ and $B_{n,2} := A_k \setminus B_{n,1}$, where $(A_{k-1})_{a_n}$ denotes the closed a_n-neighborhood of A_{k-1}. Since $A_k = B_{n,1} \cup B_{n,2}$, either the following (1) or (2) holds:

(1) $\nu_{n,k}(B_{n,1}) \geq m_n/2$ for any sufficiently large $n \in \mathbb{N}$.

(2) $\nu_{n,k}(B_{n,2}) \geq m_n/2$ for infinitely many $n \in \mathbb{N}$.

We first consider the case (2). We denote by $\mathcal{C}_n$ the set of all connected components of the set $B_{n,2}$.

Claim 3.4. *There exists $C_n \in \mathcal{C}_n$ such that $\nu_{n,k}(C_n) \geq m_n/6$.*

Proof. If $\nu_{n,k}(C) < m_n/6$ for all $C \in \mathcal{C}_n$, then there exists $\mathcal{C}'_n \subseteq \mathcal{C}_n$ such that

$$\frac{m_n}{6} \leq \nu_{n,k}\Big(\bigcup_{C' \in \mathcal{C}'_n} C'\Big) < \frac{m_n}{3}$$

because of $\nu_{n,k}(B_{n,2}) \geq m_n/2$. Putting $\mathcal{C}''_n := \mathcal{C}_n \setminus \mathcal{C}'_n$, we therefore obtain

$$2a_n \leq d_{\ell^q}\Big(\bigcup_{C' \in \mathcal{C}'_n} C', \bigcup_{C'' \in \mathcal{C}''_n} C''\Big) \leq \operatorname{Sep}\Big(\nu_{n,k}; \frac{m_n}{6}, \frac{m_n}{6}\Big) < a_n,$$

which is a contradiction. This completes the proof of the claim. □

Claim 3.5. *Putting $D_n := (C_n)_{\operatorname{Sep}(\nu_{n,k}; m_n/6, \kappa/2)} \cap A_k$, we have $\nu_{n,k}(D_n) \geq m_n - \kappa/2$.*

Proof. Take any $\delta > 0$. Supposing that $\nu_{n,k}((D_n)_\delta) < m_n - \kappa/2$, by Claim 3.4, we get

$$\operatorname{Sep}\Big(\nu_{n,k}; \frac{m_n}{6}, \frac{\kappa}{2}\Big) < d_{\ell^q}(C_n, A_k \setminus (D_n)_\delta) \leq \operatorname{Sep}\Big(\nu_{n,k}; \frac{m_n}{6}, \frac{\kappa}{2}\Big),$$

which is a contradiction. This proves that $\nu_{n,k}((D_n)_\delta) \geq m_n - \kappa$ for any $\delta > 0$. Tending $\delta \to 0$, we obtain the claim. □

Observe that D_n is isometrically embbeded into the ℓ^q-space $(\mathbb{R}^k, d_{\ell^q})$. Combining Lemma 2.7 and Claim 3.5, we therefore obtain

$$\begin{aligned} \operatorname{diam}(\nu_{n,k}, m_n - \kappa) &\leq \operatorname{diam}(\nu_{n,k}|_{D_n}, m_n - \kappa) \\ &\leq \operatorname{diam}\Big(\nu_{n,k}|_{D_n}, \nu_{n,k}(D_n) - \frac{\kappa}{2}\Big) \\ &\leq k^{1/q} \operatorname{Sep}\Big(\nu_{n,k}|_{D_n}; \frac{\kappa}{4k}, \frac{\kappa}{4k}\Big) \\ &\leq k^{1/q} \operatorname{Sep}\Big(\nu_{n,k}; \frac{\kappa}{4k}, \frac{\kappa}{4k}\Big) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies (3.2).

We next consider the case (1). Putting $b_n := a_n + \operatorname{Sep}(\nu_{n,k}; m_n/2, \kappa/2)$, as in the proof of Claim 3.5, we get

$$\nu_{n,k}((A_{k-1})_{b_n} \cap A_k) = \nu_{n,k}((B_{n,1})_{\operatorname{Sep}(\nu_{n,k}; m_n/2, \kappa/2)} \cap A_k) \geq m_n - \frac{\kappa}{2}.$$

Note that there exists a Borel measurable map $f_n : (A_{k-1})_{b_n} \cap A_k \to A_{k-1}$ such that

$$d_{\ell^q}(x, f_n(x)) = \min\{ d_{\ell^q}(x, y) \mid y \in A_{k-1}\} \leq b_n \tag{3.4}$$

for any $x \in (A_{k-1})_{b_n} \cap A_k$. Put $\nu_{n,k-1} := (f_n)_*(\nu_{n,k}|_{(A_{k-1})_{b_n} \cap A_k})$. An easy calculation proves that

$$\operatorname{Sep}(\nu_{n,k-1}; \kappa_1, \kappa_2) \leq \operatorname{Sep}(\nu_{n,k}; \kappa_1, \kappa_2) + 2b_n$$

for any $\kappa_1, \kappa_2 > 0$. By this and the property (3.2) for $\nu_{n,k}$, the measures $\nu_{n,k-1}$ on A_{k-1} satisfy that

$$\lim_{n\to\infty} \operatorname{Sep}(\nu_{n,k-1}; \kappa_1, \kappa_2) = 0$$

for any $\kappa_1, \kappa_2 > 0$. By the assumption of the induction, we therefore get

$$\lim_{n\to\infty} \operatorname{diam}\Big(\nu_{n,k-1}, \nu_{n,k-1}(A_{k-1}) - \frac{\kappa}{2}\Big) = 0$$

for any $\kappa > 0$. By using (3.4), we finally obtain

$$\begin{aligned}\operatorname{diam}(\nu_{n,k}, m_n - \kappa) \leq\ & \operatorname{diam}(\nu_{n,k-1}, m_n - \kappa) + 2b_n \\ \leq\ & \operatorname{diam}(\nu_{n,k-1}, \nu_{n,k-1}(A_{k-1}) - \kappa/2) + 2b_n \to 0 \text{ as } n \to \infty.\end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 1.1. Lemma 2.8 directly implies the sufficiency of Theorem 1.1. We shall prove the converse. Let $\{f_n : X_n \to (B^\infty_{\ell^p}, d_{\ell^q})\}_{n=1}^\infty$ be any sequence of 1-Lipschitz maps. Given an arbitrary $\varepsilon > 0$, we shall prove that

$$\operatorname{diam}((f_n)_*(\mu_{X_n}), m_{X_n} - \kappa) \leq 2\varepsilon$$

for any $\kappa > 0$ and any sufficiently large $n \in \mathbb{N}$. Put $k := k(\varepsilon)$ and $\nu_{n,k} := (F_{\infty,\varepsilon} \circ f_n)_*(\mu_{X_n})$. Since

$$\operatorname{diam}((f_n)_*(\mu_{X_n}), m_{X_n} - \kappa) \leq \operatorname{diam}(\nu_{n,k}, m_{X_n} - \kappa) + \varepsilon$$

by (3.1), it suffices to prove that

$$\lim_{n\to\infty} \operatorname{diam}(\nu_{n,k}, m_{X_n} - \kappa) = 0. \tag{3.5}$$

Since Lemma 2.2 together with Corollary 2.6 and Lemma 3.2 implies that

$$\operatorname{Sep}(\nu_{n,k}; \kappa_1, \kappa_2) \leq (1 + k(\varepsilon)^{1/q}) \operatorname{Sep}(X_n; \kappa_1, \kappa_2) \to 0 \text{ as } n \to \infty$$

for any $\kappa_1, \kappa_2 > 0$, by virtue of Lemma 3.3, we obtain (3.5). This completes the proof. □

4. Case of $1 \leq q \leq p \leq +\infty$

For an mm-space X, we define the *concentration function* $\alpha_X : (0, +\infty) \to \mathbb{R}$ as the supremum of $\mu_X(X \setminus A_{+r})$, where A runs over all Borel subsets of X with $\mu_X(A) \geq m_X/2$ and A_{+r} is an open r-neighborhood of A.

Lemma 4.1 (cf. [3, Corollary 2.6]). *A sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces is a Lévy family if and only if $\lim_{n\to\infty} \alpha_{X_n}(r) = 0$ for any $r > 0$.*

Let $p \geq 1$. We shall consider the ℓ_p^n-sphere $\mathbb{S}^n_{\ell^p} := \{(x_i)_{i=1}^n \in \mathbb{R}^n \mid \sum_{i=1}^\infty |x_i|^p = 1\}$. We denote by $\mu_{n,p}$ the cone measure and $\nu_{n,p}$ the surface measure on $\mathbb{S}^n_{\ell^p}$ normalized as $\mu_{n,p}(\mathbb{S}^n_{\ell^p}) = \nu_{n,p}(\mathbb{S}^n_{\ell^p}) = 1$. In other words, for any Borel subset $A \subseteq \mathbb{S}^n_{\ell^p}$, we put

$$\mu_{n,p}(A) := \frac{1}{\mathcal{L}(B^n_{\ell^p})} \cdot \mathcal{L}(\{tx \mid x \in A \text{ and } 0 \leq t \leq 1\}),$$

where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^n$.

By the works of G. Schechtman and J. Zinn [27, Theorems 3.1 and 4.1] and R. Latała and J. O. Wojtaszczyk [15, Theorem 5.31], we obtain

$$\alpha_{(\mathbb{S}^n_{\ell^p}, d_{\ell^2}, \mu_{n,p})}(r) \leq C\exp(-cnr^{\min\{2,p\}}). \tag{4.1}$$

This inequality for $p \geq 2$ is also mentioned by A. Naor in [23, Introduction] (see also [15, Proposition 5.21]).

Lemma 4.2. *Let $1 \leq q \leq p \leq +\infty$. Then, we have*

$$\alpha_{(\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \mu_{n,p})}(r) \leq C\exp(-cn^{1+(1/2-1/q)\min\{2,p\}} r^{\min\{2,p\}}) \text{ if } q < 2$$

and

$$\alpha_{(\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \mu_{n,p})}(r) \leq C\exp(-cnr^{\min\{2,p\}}) \text{ if } q \geq 2.$$

Proof. If $q < 2$, by $d_{\ell^q}(x,y) \leq n^{1/q-1/2}\, d_{\ell^2}(x,y)$, we then have

$$\alpha_{(\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \mu_{n,p})}(r) \leq \alpha_{(\mathbb{S}^n_{\ell^p}, d_{\ell^2}, \mu_{n,p})}(n^{1/2-1/q} r) \leq C\exp(-cn^{1+(1/2-1/q)\min\{2,p\}} r^{\min\{2,p\}}).$$

If $q \geq 2$, by $d_{\ell^q}(x,y) \leq d_{\ell^2}(x,y)$, we then obtain

$$\alpha_{(\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \mu_{n,p})}(r) \leq \alpha_{(\mathbb{S}^n_{\ell^p}, d_{\ell^2}, \mu_{n,p})}(r) \leq C\exp(-cnr^{\min\{2,p\}}).$$

This completes the proof. □

Corollary 4.3. *The sequences $\{(\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \mu_{n,p})\}_{n=1}^{\infty}$ and $\{(\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \nu_{n,p})\}_{n=1}^{\infty}$ are both Lévy families for $1 \leq q \leq p \leq +\infty$.*

Proof. Since $1+(1/2-1/q)\min\{2,p\} > 0$, by Lemmas 4.1 and 4.2, the sequence $\{(\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \mu_{n,p})\}_{n=1}^{\infty}$ is a Lévy family. By virtue of [23, Theorem 6], the sequence $\{\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \nu_{n,p})\}_{n=1}^{\infty}$ is also a Lévy family. This completes the proof. □

Proposition 4.4. *Let $1 \leq q \leq p \leq +\infty$. Then, for any κ with $0 < \kappa < 1/2$, we have*

$$\mathrm{ObsDiam}_{(B^{\infty}_{\ell^p}, d_{\ell^q})}((\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \mu); -\kappa) \geq 2,$$

where $\mu = \mu_{n,p}$ or $\mu = \nu_{n,p}$.

Proof. Let $A \subseteq \mathbb{S}^n_{\ell^p}$ be a Borel subset such that $\mu(A) \geq 1-\kappa$. Since $\mu(A) = \mu(-A) > 1/2$, we have $\mu(A \cap (-A)) > 0$. Hence, there exists $x \in A$ such that $-x \in A$. Since $\operatorname{diam} A \geq d_{\ell^q}(x,-x) \geq d_{\ell^p}(x,-x) = 2$, we obtain $\operatorname{diam}(\mu, 1-\kappa) = 2$. Since the inclusion map from the space $(\mathbb{S}^n_{\ell^p}, d_{\ell^q})$ to the space $(B^{\infty}_{\ell^p}, d_{\ell^q})$ is 1-Lipschitz, we obtain the conclusion. This completes the proof. □

Combining Corollary 4.3 with Proposition 4.4, we obtain an example of a Lévy family which does not satisfy (1.1) in the case of $1 \leq q \leq p \leq +\infty$.

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References

[1] K. Funano, *Asymptotics of mm-spaces*, Doctoral thesis, in preparation.

[2] K. Funano, *Central and L^p-concentration of 1-Lipschitz maps into $\mathbb{R}$-trees*, to appear in J. Math. Soc. Japan.

[3] K. Funano, *Concentration of maps and group action*, preprint, available online at "http://front.math.ucdavis.edu/0807.3210", 2008.

[4] K. Funano, *Observable concentration of mm-spaces into spaces with doubling measures*, Geom. Dedicata **127**, 49–56, 2007.

[5] K. Funano, *Observable concentration of mm-spaces into nonpositively curved manifolds*, preprint, available online at "http://front.math.ucdavis.edu/0701.5535", 2007.

[6] T. Giordano and V. Pestov, *Some extremely amenable groups*, C. R. Acad. Sci. Paris, Sér. I **334**, No. 4, 273–278, 2002.

[7] T. Giordano and V. Pestov, *Some extremely amenable groups related to operator algebras and ergodic theory*, J. Inst. Math. Jussieu **6**, no. 2, 279–315, 2007.

[8] S. Glasner, *On minimal actions of Polish groups*, Top. Appl. **85**, 119–125, 1998.

[9] A. Gournay, *Width of ℓ^p-balls*, preprint, available online at "http://front.math.ucdavis.edu/0711.3081", 2007.

[10] M. Gromov and V. D. Milman, *A topological application of the isoperimetric inequality*, Amer. J. Math. **105**, no. 4, 843–854, 1983.

[11] M. Gromov, CAT(κ)*-spaces: construction and concentration*, (Russian summary) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 280, Geom. i Topol. **7**, 100–140, 299–300, 2001; translation in J. Math. Sci. (N. Y.) **119**, no. 2, 178–200, 2004.

[12] M. Gromov, *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal., **13**, no. 1, 178–215, 2003.

[13] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by Sean Michael Bates. Progress in Mathematics, **152**. Birkhäuser Boston, Inc., Boston, MA, 1999.

[14] M. Gromov, *Topological invariants of dynamical systems and spaces of holomorphic maps*, I. Math. Phys. Anal. Geom. **2**, no. 4, 323–415, 1999.

[15] R. Latała and J. O. Wojtaszczyk, *On the infimum convolution inequality*, preprint, available online at "http://arxiv.org/abs/0801.4036", 2008.

[16] M. Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, **89**. American Mathematical Society, Providence, RI, 2001.

[17] V. D. Milman, *A certain property of functions defined on infinite-dimensional manifolds*, (Russian) Dokl. Akad. Nauk SSSR **200**, 781–784, 1971.

[18] V. D. Milman, *A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies*, (Russian) Funkcional. Anal. i Priložen. **5**, no. 4, 28–37, 1971.

[19] V. D. Milman, *Asymptotic properties of functions of several variables that are defined on homogeneous spaces*, Soviet Math. Dokl. **12**, 1277–1281, 1971; translated from Dokl. Akad. Nauk SSSR **199** , 1247–1250, 1971(Russian).

[20] V. D. Milman, *Diameter of a minimal invariant subset of equivariant subset of equivariant Lipschitz actions on compact subsets of* $\mathbb{R}^k$, Grometric Aspects of Functional Analysis, Israel Seminar, 1985–1986. Lecture Notes in Math. **1267**, 13–20, Springer, 1987.

[21] V. D. Milman, *The heritage of P. Lévy in geometrical functional analysis*, Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987). Ast érisque No. **157-158**, 273–301, 1988.

[22] V. D. Milman, G. Schechtman, *Asymptotic theory of finite-dimensional normed spaces. With an appendix by M. Gromov*, Lecture Notes in Mathematics, **1200**. Springer-Verlag, Berlin, 1986.

[23] A. Naor, *The surface measure and cone measure on the sphere of* ℓ_p^n, Trans. Amer. Math. Soc. **359**, 1045–1079, 2007.

[24] V. Pestov, *Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman phenomenon,* Revised edition of Dynamics of infinite-dimensional groups and Ramsey-type phenomena [Inst. Mat. Pura. Apl. (IMPA), Rio de Janeiro, 2005; MR2164572].University Lecture Series, **40**. American Mathematical Society, Providence, RI, 2006.

[25] V. Pestov, *Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups.* - Israel Journal of Mathematics **127**, 317–358, 2002. *Corrigendum*, ibid., **145**, 375–379, 2005.

[26] V. Pestov, *The isometry groups of the Urysohn metric space as a Lévy group*, Topology Appl. **154**, no. 10, 2173–2184, 2007.

[27] G. Schechtman and J. Zinn, *Concentration on the ℓ_p^n Ball*, Geometric aspects of functional analysis, 245–256, Lecture Notes in Math., **1745**, Springer, Berlin, 2000.

[28] M. Tsukamoto, *Macroscopic dimension of the ℓ^p-ball with respect to the ℓ^q-norm*, to appear in J. Math. Kyoto Univ.

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